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GRADED IDENTITIES FOR T-PRIME ALGEBRAS OVER FIELDS OF POSITIVE CHARACTERISTIC

BY

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ABSTRACT

In this paper we study 2-graded polynomial identities. We describe bases of these identities satisfied by the matrix algebra of order two $M_2(K)$, by the algebra $M_{1,1}(G)$, and by the algebra $G \otimes_K G$. Here K is an arbitrary infinite field of characteristic not 2, G stands for the Grassmann (or exterior) algebra of an infinite dimensional vector space over K, and $M_{1,1}(G)$ is the algebra of all 2×2 matrices over G whose entries on the main diagonal are even elements of G, and those on the second diagonal are odd elements of G. The gradings on these three algebras are supposed to be the standard ones.

It turns out that the graded identities of these three algebras are closely related, and furthermore $M_{1,1}(G)$ and $G \otimes G$ satisfy the same 2-graded identities provided that $\operatorname{char} K = 0$. When $\operatorname{char} K = p > 2$, then the algebra $G \otimes G$ satisfies some additional 2-graded identities that are not identities for $M_{1,1}(G)$. The methods used in the proofs are based on appropriate constructions for the corresponding relatively free algebras, on combinatorial properties of permutations, and on a version of Specht's commutator reduction. We hope that this paper is a step towards the description of the ordinary identities satisfied by the algebras $G \otimes G$ and $M_{1,1}(G)$ over an infinite field of positive characteristic. Note that in characteristic 0 such a description was given in [12] and in [10].

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Introduction

The $\mathbb{Z}_2$-graded algebras and the polynomial identities satisfied by such algebras are an important ingredient in the structure theory of PI algebras; see, for example, [10], [1]. It is known that every non-trivial verbally prime T-ideal in the free associative algebra over a field K, $\operatorname{char} K = 0$, is the ideal of identities in some of the following algebras. First come the matrix algebras $M_n(K)$ of order n, then the matrix algebras $M_n(G)$ of order n over the Grassmann algebra G of an infinite dimensional K-vector space. The third type of verbally prime T-ideals consists of the identities satisfied by the algebras $M_{k,l}(G)$. These algebras are subalgebras of $M_n(G)$ for $n = k + l$, they consist of the matrices of type $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \in M_k(G_0)$, $D \in M_l(G_0)$, $k \geq l$, and $B \in M_{k\times l}(G_1)$, $C \in M_{l\times k}(G_1)$. Here we consider G_0 as the subalgebra of G generated by all elements having even length, and G_1 the subspace of G spanned by the elements of odd length, $G = G_0 \oplus G_1$. By length of an element g of G we mean the number t if $g = e_{i_1} e_{i_2} \cdots e_{i_t}$ where $i_1 < i_2 < \cdots < i_t$ and $\{e_i \mid i \in \mathbb{N}\}$ are a K-basis for the underlying vector space, with multiplication $e_i e_j = -e_j e_i$, $i \neq j$, and $e_i^2 = 0$. Even in characteristic 0 little is known about the concrete polynomial identities satisfied by these algebras except for the ones satisfied by G, $M_2(K)$, and $G \otimes G$ and $M_{1,1}(G)$. The picture becomes even more unclear when $\operatorname{char} K = p > 0$. In the latter case a basis of the identities for the algebra G is known ([4], and if unavailable, see for a brief survey [8]). Recently, a finite basis of the identities of $M_2(K)$ was described when $\operatorname{char} K = p > 2$ and K is infinite ([11]).

Hence one is led to study other types of polynomial identities such as weak identities, identities with trace, graded and/or with involution etc. Thus, for example, the trace identities of the algebras $M_n(K)$ and of $M_{k,l}$ over a field of characteristic 0 were described by Razmyslov and by Procesi; see [14], [13]. The 2-graded identities of the algebras $M_2(K)$ and $M_{1,1}(G)$ over a field of characteristic 0 were described in [5], and the n-graded identities of $M_n(K)$ in [18]. Let us mention that the information about the graded identities in $M_2(K)$ obtained by O. M. Di Vincenzo in [5] allowed him to give a new proof of the "standard" fact that the algebras $G \otimes G$ and $M_{1,1}(G)$ satisfy the same polynomial identities when the base field is of characteristic 0. (Note that otherwise this fact follows from Kemer's classification, see [10].) The interest in the study of 2-graded identities in algebras over a field of characteristic 0 is justified by the relationship between the graded and ordinary polynomial identities which is one of the key components in the structure theory of T-ideals developed by A. Kemer ([10], Theorem 1.1, [1], Theorem 7). This relationship is the following. If A is an (associative) algebra over a field of characteristic 0, then it satisfies the same identities as the algebra $B_0 \otimes G_0 \oplus B_1 \otimes G_1$ does where $B = B_0 \oplus B_1$ is a finitely generated 2-graded algebra and $G = G_0 \oplus G_1$ is the Grassmann algebra with its natural grading.

Although in positive characteristic there does not exist such a theorem, the graded identities are still of interest; see, for example, [2, 3].

In this paper we determine bases of the 2-graded polynomial identities satisfied by the algebras $M_2(K)$, $M_{1,1}(G)$ and $G \otimes G$, over an infinite field K of characteristic $p \neq 2$. Furthermore, we show that the last two algebras satisfy the same 2-graded identities when $\operatorname{char} K = 0$, and when $\operatorname{char} K = p > 2$ the latter algebra satisfies some more 2-graded identities. In order to obtain these results we construct appropriate models for the respective relatively free (2-graded) algebras. The description of the 2-graded identities satisfied by $M_2(K)$ uses some ideas from [5]. Furthermore, we make use of various combinatorial techniques and of certain graded variant of the Specht's reduction to commutator polynomial identities. The bases for the 2-graded identities of the algebras $M_2(K)$ and of $M_{1,1}(G)$ are exactly the same as in the case of characteristic 0. (Note that the bases of the ordinary identities satisfied by the algebras $M_{1,1}(G)$ and $G \otimes G$ over infinite fields of characteristic $p > 2$ are still unknown.)

The paper is organized as follows. The next section contains mainly definitions and preliminary information needed to follow the exposition. Section 2 deals with the 2-graded identities of the algebra $M_2(K)$. We prove that the graded identities of $M_2(K)$ follow from two identities, namely from $y_1y_2 - y_2y_1$ and $z_1z_2z_3 - z_3z_2z_1$ for y_i even and z_i odd variables. In Section 3 we describe the 2-graded identities satisfied by $M_{1,1}(G)$, and finally in Section 4 we study these of $G \otimes G$. It turns out that the graded identities of $M_{1,1}(G)$ are consequences of $y_1y_2 - y_2y_1$ and of $z_1z_2z_3 + z_3z_2z_1$. These of $G \otimes G$ follow from $y_1y_2 - y_2y_1$, $z_1z_2z_3 + z_3z_2z_1$, and if $\operatorname{char} K = p > 2$ one adds the identity $y_1^p z_1 - z_1 y_1^p$. Note that the last identity is not satisfied by $M_{1,1}(G)$.

We hope that this paper will be a step towards the description of the ordinary identities satisfied by the algebras $M_{1,1}(G)$ and $G \otimes G$ over fields of positive characteristic, and thus will contribute to better understanding of the structure of the T-ideals over such fields.

1. Preliminaries

Throughout, we consider unitary associative algebras over a fixed infinite field K of characteristic $p \neq 2$; all tensor products are over K (and hence we shall omit the subscript for the tensor products). Let $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$

be two disjoint sets of symbols and $X = Y \cup Z$. Denote by $K(X)$ the free associative algebra that is freely generated over K by the set X. An algebra A is called 2-graded (or superalgebra) if $A = A_0 \oplus A_1$ where A_0 is a subalgebra of A and A_1 a subspace, and furthermore, $A_i A_j \subseteq A_{i+j}$ where we sum the indices modulo 2. Let G be the Grassmann (or exterior) algebra of a vector space V with a basis $e_1, e_2, \ldots$. Then G is spanned by 1 and by the products $e_{i_1} e_{i_2} \cdots e_{i_k}$ where $i_1 < i_2 < \cdots < i_k$, $k = 1, 2, \ldots$, and the multiplication in G is induced by $e_i e_j = -e_j e_i$ when $i \neq j$, and $e_i^2 = 0$. Hence $G = G_0 \oplus G_1$, where G_0 is the subspace of G spanned by all monomials in the e_i's of even length, and G_1 is the subspace spanned by all elements of odd length. Therefore G becomes a 2-graded algebra, and G_0 is the centre of G. Obviously one has $ab = -ba$ for all a, $b \in G_1$. One may introduce various gradings on an algebra. We fix the following gradings on $M_2(K)$, $M_{1,1}(G)$ and on $G \otimes G$, and we shall refer to them as to the standard gradings (or simply gradings).

These are the following:

$$M_2(K) = A_0 \oplus A_1, \quad A_0 = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\}, \quad A_1 = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\},$$

for a, b, c, $d \in K$;

$$M_{1,1}(G) = B_0 \oplus B_1, \quad B_0 = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\}, \quad B_1 = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\},$$

for a, $d \in G_0$, b, $c \in G_1$;

$$G \otimes G = (G_0 \otimes G_0 \oplus G_1 \otimes G_1) \oplus (G_0 \otimes G_1 \oplus G_1 \otimes G_0).$$

Let f be a monomial in the free algebra $K(X)$. We say that f is even if it contains an even number of entries from Z, i.e., if its degree with respect to the symbols in Z is even. Otherwise f is called odd. The span of all even (odd) monomials is denoted by $K(X)_0$ (respectively by $K(X)_1$). Therefore $K(X) = K(X)_0 \oplus K(X)_1$ becomes a 2-graded algebra. It is the free 2-graded algebra, and its elements will be called polynomials. If $A = A_0 \oplus A_1$ is 2-graded algebra and $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in K(X)$, then f is graded identity for A if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ for all $a_i \in A_0$, $b_j \in A_1$.

The set $T_2(A)$ of all graded identities of A is an ideal of $K(X)$. It is called the T_2-ideal of A. It can be easily verified that $T_2(A)$ is closed under 2-graded endomorphisms of A (i.e., under endomorphisms that preserve the even and the odd parts of A). If $g \in K(X)$ we shall say that g is T_2-consequence of f (or that g follows from f as graded identity) if g belongs to the T_2-ideal generated in $K(X)$ by f. Here and in what follows, T_2-ideal means the ideal of 2-graded identities satisfied by some 2-graded algebra.

The field K is infinite. Hence every polynomial $f \in K(X)$ is equivalent as graded identity to a finite collection of multihomogeneous graded identities. Therefore we may and shall consider multihomogeneous polynomials only.

For a, $b \in K(X)$ we denote as $[a,b] = ab - ba$ the commutator of a and b. The higher commutators are assumed left normed, i.e., we define them inductively by

$$[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n], \quad n \geq 3.$$

Denote as $B(X)$ the subalgebra of $K(X)$ generated by all commutators $[x_{i_1}, x_{i_2}, \dots, x_{i_n}]$, $n = 2, 3, \dots$. It is well known that if T is a T-ideal in $K(X)$ (i.e., T is the ideal of identities of some unitary algebra, or equivalently, T is stable under all endomorphisms of the algebra $K(X)$), then T is generated as T-ideal by its elements from $B(X)$. In other words $T = \langle T \cap B(X) \rangle^T$. This fact was first observed by W. Specht [17]. Though in [17] this result was proved for multilinear polynomials and, formally speaking, in characteristic 0 only, the same method of proof yields the result in every characteristic provided the field is infinite; see, for example, [7], pp. 42–43, Proposition 4.3.3. We need a modification of this statement in the case of graded identities.

LEMMA 1: *If $f(y_1, \dots, y_m, z_1, \dots, z_n) \in K(X)$ is a multihomogeneous polynomial, then it is equivalent as graded identity to a finite collection of graded identities such that the even variables y_1, $\dots$, y_m appear in any one of them in commutators only.*

Proof: The proof repeats verbatim that of [7], Proposition 4.3.3. Note that one can substitute $x_i + 1$ for x_i only in the cases $x_i \in Y$ since $1 \in K(X)$ belongs to $K(X)_0$.

We denote as $B_2 = B_2(X)$ the set of the polynomials f in $K(X)$ such that every even variable y_i is in commutators in the expansion of f.

COROLLARY 2: *Let T be T_2-ideal in $K(X)$. Then T is generated as T_2-ideal by the set $T \cap B_2$.*

2. The graded identities of $M_2(K)$

In this section we describe the ideal of the graded identities satisfied by the algebra $M_2(K)$ with respect to the standard grading. Our method is similar to that of [5].

Let $\mathrm{Gen}(M_2(K))$ be the generic matrix algebra of order 2 generated by some countable set of matrices

$$A_i = \begin{pmatrix} x_i^{(1)} & x_i^{(2)} \\ x_i^{(3)} & x_i^{(4)} \end{pmatrix}.$$

Here $x_i^{(j)}$, $i \geq 1$, $1 \leq j \leq 4$ are commuting variables. It is well known that $\mathrm{Gen}(M_2(K))$ is isomorphic to the relatively free algebra in the variety of algebras determined by the matrix algebra of order two.

Denote by $F(M_2(K))$ the subalgebra of $M_2(K[y_i^{(j)}, z_i^{(j)} \mid i \geq 1, j = 1,2])$ generated by the matrices

$$A_i = \begin{pmatrix} y_i^{(1)} & 0 \\ 0 & y_i^{(2)} \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & z_i^{(1)} \\ z_i^{(2)} & 0 \end{pmatrix}$$

where $K[y_i^{(j)}, z_i^{(j)} \mid i \geq 1, j = 1,2]$ is the polynomial algebra generated by the variables $y_i^{(j)}$, $z_i^{(j)}$. The algebra $F(M_2(K))$ possesses 2-grading in a natural manner. Namely its matrices of the type $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ form the even part, while those of the type $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ form the odd part.

Denote by $T_2(M_2) = T_2(M_2(K))$ the ideal of the graded identities for $M_2(K)$.

LEMMA 3: *The relatively free 2-graded algebra $K\langle X\rangle/T_2(M_2)$ is isomorphic to the algebra $F(M_2(K))$.*

Proof: The proof is analogous to that for the generic matrices. Denote by e_{11}, e_{12}, e_{21}, e_{22} the usual basis of the vector space $M_2(K)$; then $e_{ij}e_{kl} = \delta_{jk}e_{il}$, where $\delta_{jk} = 0$ if $j \neq k$ and $\delta jj = 1$. Put $Y_i = y_i^{(1)}e_{11} + y_i^{(2)}e_{22}$ and $Z_i = z_i^{(1)}e_{12} + z_i^{(2)}e_{21}$. Then define a homomorphism Φ: $K\langle X\rangle \to F(M_2(K))$ by $\Phi(y_i) = Y_i$ and $\Phi(z_i) = Z_i$. An easy calculation shows that $\ker\Phi = T_2(M_2(K))$ and Φ is an isomorphism, as required.

Thus we shall work in the 2-graded algebra $F(M_2(K))$ instead of $K\langle X\rangle/T_2(M_2)$. The following lemma is a well-known fact (and easy to deduce as well).

LEMMA 4: *The graded identities $y_1y_2 - y_2y_1$ and $z_1z_2z_3 - z_3z_2z_1$ belong to the T_2-ideal $T_2(M_2)$.*

PROPOSITION 5: (a) *If $g \in K\langle X\rangle_0$ then $y_ig - gy_i \in T_2(M_2(K))$ and this graded identity follows from the two identities of the previous lemma;*

(b) *The relatively free 2-graded algebra $K(X)/T_2(M_2)$ is spanned over K by 1 and by the following monomials:*

$$y_{a_1}y_{a_2}\cdots y_{a_k},$$
$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}y_{b_1}y_{b_2}\cdots y_{b_l},$$
$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}\widehat{z_{d_m}},$$
$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}y_{b_1}y_{b_2}\cdots y_{b_l}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}\widehat{z_{d_m}}.$$

Here $a_1 \leq a_2 \leq \cdots \leq a_k$, $b_1 \leq b_2 \leq \cdots \leq b_l$, $c_1 \leq c_2 \leq \cdots \leq c_m$ and $d_1 \leq d_2 \leq \cdots \leq d_m$, $k \geq 0$, $l \geq 0$, $m \geq 0$. In the second type of monomials $k + l \geq 1$; in the fourth, if $k = l = 0$ its degree is ≥ 2, and the "hat" over a variable means that it can be missing.

(c) *The monomials of* (b) *are linearly independent modulo the T_2-ideal $T_2(M_2)$.*

Proof: The proof uses ideas from [5], Lemma 2. The statement of (a) is evident since g is an even element and it commutes with y_i modulo the graded identities of $M_2(K)$.

(b) Using (a), one obtains that every monomial from $K(X)/T_2(M_2)$ is a linear combination of monomials of the form

$$h_1(y)\widehat{z_{e_1}}h_2(y)z_{e_2}z_{e_3}\cdots z_{e_s}$$

where $h_1(y)$ and $h_2(y)$ are monomials in the y_i's. Due to the identity $y_1y_2 - y_2y_1$, we may suppose that the indices of the variables in $h_1(y)$ and in $h_2(y)$ increase (with possible repetitions). Now the identity $z_1z_2z_3 - z_3z_2z_1$ yields the rest of the statement. Note that if $e_1 > e_3$ in a monomial, then we use the fact that $z_{e_1}(h_2(y)z_{e_2})z_{e_3} - z_{e_3}(h_2(y)z_{e_2})z_{e_1}$ belongs to $T_2(M_2)$.

(c) In order to prove the linear independency of the above monomials, it is sufficient to prove it for every set of multihomogeneous monomials (of the same multidegree). It is convenient to make the calculations in the "generic" 2-graded algebra. Set $y_i = A_i$, $z_i = B_i$ where A_i and B_i are the matrices introduced at the beginning of this section. Then

$$A_{a_1}A_{a_2}\cdots A_{a_k} = y_{a_1}^{(1)}y_{a_2}^{(1)}\cdots y_{a_k}^{(1)}e_{11} + y_{a_1}^{(2)}y_{a_2}^{(2)}\cdots y_{a_k}^{(2)}e_{22}$$

and that shows the independency of the monomials of the first type. Analogously one obtains

$$\begin{aligned}A_{a_1}\ldots A_{a_k}B_{c_1}A_{b_1}\ldots A_{b_l} =& y_{a_1}^{(1)}\ldots y_{a_k}^{(1)}z_{c_1}^{(1)}y_{b_1}^{(2)}\ldots y_{b_l}^{(2)}e_{12}\\ &+ y_{a_1}^{(2)}\ldots y_{a_k}^{(2)}z_{c_1}^{(2)}y_{c_1}^{(1)}\ldots y_{c_l}^{(1)}e_{21}.\end{aligned}$$

Thus one can "reconstruct" uniquely the monomial by the above expression. Therefore if a multihomogeneous polynomial in these monomials equals 0 in the relatively free 2-graded algebra $K(X)/T_2(M_2)$, all of its coefficients must be zero.

Finally, we consider the monomials of the last type. Choose a monomial of such type that belongs to $K(X)_0$. In other words, there is an even number of z_i's in the expansion of the monomial. In analogy with the above case, one evaluates the monomial

$$M = y_{a_1} y_{a_2} \cdots y_{a_k} z_{c_1} y_{b_1} y_{b_2} \cdots y_{b_l} z_{d_1} z_{c_2} z_{d_2} \cdots z_{c_m} z_{d_m}$$

on the matrices A_i and B_i, and one obtains the expression

$$\begin{gathered} y_{a_1}^{(1)} \cdots y_{a_k}^{(1)} z_{c_1}^{(1)} y_{b_1}^{(2)} \cdots y_{b_l}^{(2)} z_{d_1}^{(2)} z_{c_2}^{(1)} z_{d_2}^{(2)} \cdots z_{c_m}^{(1)} e_{11} \\ + y_{a_1}^{(2)} \cdots y_{a_k}^{(2)} z_{c_1}^{(2)} y_{b_1}^{(1)} \cdots y_{b_l}^{(1)} z_{d_1}^{(1)} z_{c_2}^{(2)} z_{d_2}^{(1)} \cdots z_{c_m}^{(2)} e_{22}. \end{gathered}$$

One reconstructs uniquely from this expression the monomial M. Suppose that a multihomogeneous polynomial whose monomials have the same multidegree as the one of M, equals 0 in the relatively free algebra $K(X)/T_2(M_2)$. Then one obtains a linear combination of matrices of the above type, and this is possible only if all its coefficients equal 0.

The remaining case when $M \in K(X)_1$ is dealt with in the same manner, and thus the proof of the proposition is complete.

COROLLARY 6 (cf. [5], Lemma 2): *Let K be an infinite field, $\operatorname{char} K = p \neq 2$. Then the 2-graded identities of the algebra $M_2(K)$ are consequences of the identities $y_1y_2 - y_2y_1$ and $z_1z_2z_3 - z_3z_2z_1$.*

Proof: The statement follows from (b) and (c) of the above proposition.

Remark: It is worth mentioning that the 2-graded identities satisfied by the matrix algebra of order two over an infinite field K, $\operatorname{char} K \neq 2$, are not very "interesting" since a basis of the ordinary identities satisfied by this algebra is known; see [11].

3. The graded identities of $M_{1,1}(G)$

In this section we treat the 2-graded identities for the algebra $M_{1,1} = M_{1,1}(G)$ in a manner similar to that of the previous section. We construct a model of the corresponding relatively free graded algebra.

We recall the definition of the free supercommutative algebra; see, for example, [2], Section 2. Let Y and Z be two disjoint infinite sets of variables, $X = Y \cup Z$, and consider the usual grading on $K(X)$ assuming the variables in Y even and those in Z odd. Then $K(X) = K(X)_0 \oplus K(X)_1$. Given $f \in K(X)_i$, $g \in K(X)_j$ we impose the relations $fg - (-1)^{ij} gf = 0$ and denote the corresponding quotient algebra by $K(Y;Z)$. Thus $K(Y;Z) = K[Y] \otimes G(Z)$ where $K[Y]$ is the ordinary polynomial commutative algebra generated by Y, and $G(Z)$ is the Grassmann algebra of the K-span of the set Z. The algebra $K(Y;Z)$ is the free supercommutative algebra, see [2], Lemma 1.

Now consider the sets

$$Y = \{y_i^{(j)} \mid i \geq 1, j = 1,2\} \quad \text{and} \quad Z = \{z_i^{(j)} \mid i \geq 1, j = 1,2\}$$

as generating sets for the free supercommutative algebra. Form the matrices

$$A_i = \begin{pmatrix} y_i^{(1)} & 0 \\ 0 & y_i^{(2)} \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & z_i^{(1)} \\ z_i^{(2)} & 0 \end{pmatrix}, \quad i = 1, 2, \ldots$$

and consider the subalgebra $\mathrm{Gen}(M_{1,1})$ of $M_2(K(Y;Z))$ generated by these matrices. It has a natural 2-grading defined as follows. The even component consists of all matrices of the type $f_{11}e_{11} + f_{22}e_{22}$ while the odd consists of $f_{12}e_{12} + f_{21}e_{21}$, $f_{ij} \in K(Y;Z)$. Note that according to the rules of multiplication one has f_{11}, $f_{22} \in K(Y;Z)_0$ and f_{12}, $f_{21} \in K(Y;Z)_1$.

It follows from [2], Theorem 2, that the K-algebra generated by the matrices $C_i = A_i + B_i$ is canonically isomorphic to the relatively free algebra of countable rank in the variety of algebras determined by $M_{1,1}$. A similar reasoning yields the following lemma. Its proof repeats verbatim that of the corresponding result for $M_2(K)$ from the previous section.

LEMMA 7: *The 2-graded algebra* $\mathrm{Gen}(M_{1,1})$ *is isomorphic to the relatively free algebra of countable rank* $F_2(M_{1,1})$ *in the variety of 2-graded algebras determined by* $M_{1,1}$.

PROPOSITION 8: *Let I be the T_2-ideal of the 2-graded identities for $M_{1,1}$. Then:*

(a) *The polynomials* $y_1y_2 - y_2y_1$, $z_1z_2z_3 + z_3z_2z_1 \in K(X)$ *belong to* I;

(b) *Consider the canonical projection* $K(X) \to K(X)/J$ *where J is the ideal of 2-graded identities generated by* $y_1y_2 - y_2y_1$ *and* $z_1z_2z_3 + z_3z_2z_1$. *Identify the variables y_i and z_i with their images under this projection. Then the monomials*

$$y_{a_1}y_{a_2}\cdots y_{a_k},$$

$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}y_{b_1}y_{b_2}\cdots y_{b_l},$$

$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}\widehat{z_{d_m}},$$

$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}y_{b_1}y_{b_2}\cdots y_{b_l}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}\widehat{z_{d_m}}$$

span $K(X)/J$. Here $a_1 \le a_2 \le \cdots \le a_k$, $b_1 \le b_2 \le \cdots \le b_l$, $c_1 < c_2 < \cdots < c_m$ and $d_1 < d_2 < \cdots < d_m$, $k \ge 0$, $l \ge 0$, $m \ge 0$. In the second type of monomials $k + l \ge 1$; in the fourth, if $k = l = 0$ its degree is ≥ 2, and the "hat" over a variable means that it can be missing.

(c) *The monomials from* (b) *are linearly independent modulo the 2-graded identities of the algebra $M_{1,1}$.*

Proof: Part (a) of the proposition is proved by direct calculation, analogously to the case of $M_2(K)$. Observe that G_0 is the centre of G, hence $y_1y_2 - y_2y_1$ is indeed 2-graded identity for $M_{1,1}$. For the second identity one uses the fact that $g_1g_2g_3 + g_3g_2g_1 = 0$ for every $g_1, g_2, g_3 \in G_1$.

(b) We proceed in the same manner as in Proposition 5 (b), and in order not to be (too) boring we omit the details. Notice only that we allow repeated entries neither in the sequence c_i nor in d_i.

(c) The proof repeats verbatim that of Proposition 5 (c) with the only difference that we work in the algebra $\mathrm{Gen}(M_{1,1})$ instead of the generic matrix algebra.

The following theorem was proved in [5], Theorem 1, in the case $\mathrm{char}\, K = 0$.

THEOREM 9: *Let K be an infinite field,* $\mathrm{char}\, K \neq 2$. *Then the 2-graded identities of the algebra $M_{1,1}(G)$ follow from the two identities $y_1y_2 - y_2y_1$ and $z_1z_2z_3 + z_3z_2z_1$.*

Proof: It is an immediate consequence of Proposition 8.

Remarks: 1. If $f \in K(X)$ is multihomogeneous and not an identity of $M_{1,1}$, and if the variable z_i occurs in the monomials of f, then $\deg_{z_i} f \le 2$. For if one has three letters z_i in a monomial, at least two of them will be in one of the sequences c_i or d_i. But due to the identity $z_1z_2z_3 + z_3z_2z_1 = 0$ such monomials vanish.

2. In [5] the above theorem was proved using properties of the involution $*$ defined on the space of multilinear polynomials; see [10], pp. 17, 18 for the precise definition. Since our field may be of positive characteristic, the multilinear elements of a T_2-ideal might not determine the T_2-ideal. In [5] the corresponding result is a direct consequence of the properties of the involution $*$ and of the description of the basis for the 2-graded identities of $M_2(K)$. This holds, since if we decompose $M_2(K) = A_0 \oplus A_1$ according to the standard grading then $M_{1,1} \cong A_0 \otimes G_0 \oplus A_1 \otimes G_1$.

3. Note that the polynomial $[y^p, z]$ does not vanish on $M_{1,1}(G)$ considered as 2-graded identity. (Here, y is an even variable, z is an odd one, and $p = \mathrm{char}\, K$.)

Choose, for instance, $y = e_{11} + 2e_{22}$, $z = ge_{12} + ge_{21}$ where e_{ij} are the matrices from the standard basis of $M_n(K)$, and $g \neq 0$ is an arbitrary element of G_1. Then $y^p = e_{11} + 2^p e_{22}$ and, since $2^p \neq 1$ in K, we have $[y^p, z] = g((1-2^p)e_{12} + (2^p - 1)e_{21}) \neq 0$.

Now we observe that we can choose another model for the algebra $\mathrm{Gen}(M_{1,1})$. Let $a_i^{(0)}$, $b_i^{(0)}$ be commuting variables and $a_i^{(1)}$, $b_i^{(1)}$ be anticommuting ones, and form the free supercommutative algebra $K(a_i^{(j)}, b_i^{(j)} \mid i = 1, 2, \ldots, j = 0, 1)$ that is freely generated by them. Consider the matrices of the form

$$C_i = a_i^{(0)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b_i^{(0)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_i = a_i^{(1)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + b_i^{(1)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and generate an algebra L by 1 and them, assuming that C_i are even, and D_i are odd elements. Then L is 2-graded algebra.

Lemma 10: *The algebra L is isomorphic to* $\mathrm{Gen}(M_{1,1})$.

Proof: Let φ: $\mathrm{Gen}(M_{1,1}) \to L$ be the homomorphism defined by $\varphi(A_i) = C_i$, $\varphi(B_i) = D_i$; then it is obviously 2-graded isomorphism.

It is immediate that the matrices $a_i^{(0)}(e_{11} + e_{22})$ commute with these of L. Denote as $B_2(L)$ the subalgebra of L that is generated by 1 and by all elements of L such that every even "variable" appears in them in commutators only. Since the matrices $a_i^{(0)}(e_{11} + e_{22})$ are central, they will disappear from the polynomials in $B_2(L)$.

Lemma 11: *The matrices* $E_i = b_i^{(0)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $D_i = \begin{pmatrix} 0 & c_i^{(1)} \\ d_i^{(1)} & 0 \end{pmatrix}$ *satisfy the following relations:*

$$E_i E_j \text{ are central}, \quad E_i E_j = E_j E_i,$$
$$E_i D_j = -D_j E_i, \quad D_i^2 D_j = -D_j D_i^2.$$

Proof: It consists of direct (and easy) verification.

Now let $B_2(M_{1,1})$ be the subalgebra of $\mathrm{Gen}(M_{1,1})$ generated by $1 = e_{11} + e_{22}$ and by the polynomials such that every even variable appears in commutators only. In other words, $B_2(M_{1,1}) = B_2(X)/(T_2(M_{1,1}) \cap B_2(X))$. Then $B_2(M_{1,1})$ is canonically isomorphic to $B_2(L)$, and we shall identify $B_2(M_{1,1})$ and $B_2(L)$.

Proposition 12 (cf. [6], Lemmas 2.2 and 2.3): *If $f \in B_2(L)$ is a multihomogeneous polynomial, then f is a linear combination of elements of the form*

$$E_{i_1}^{\alpha_1} E_{i_2}^{\alpha_2} \cdots E_{i_k}^{\alpha_k} D_{j_1}^2 D_{j_2}^2 \cdots D_{j_l}^2 g(D_{n_1}, D_{n_2}, \cdots, D_{n_m})$$

where $i_1 < i_2 < \cdots < i_k$, $\{j_1, j_2, \ldots, j_l\}$ *and* $\{n_1, n_2, \ldots, n_m\}$ *are disjoint sets,* $j_1 < j_2 < \cdots < j_l$, *and the polynomial* g *is multilinear.*

Proof: First, since $e = e_{11} + e_{22}$ is central, there will be no matrices of type $a_i^{(0)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in the expansion of f. Then one uses the preceding lemma and obtains that

$$f = E_{i_1}^{\alpha_1} E_{i_2}^{\alpha_2} \cdots E_{i_k}^{\alpha_k} h(D_1, D_2, \ldots, D_t)$$

where h is a multihomogeneous polynomial. If the degree of h in some D_i were larger than 2, then h would be 2-graded identity for $M_{1,1}$; see the Remark above. Hence we may suppose that the degree of h in any one of its variables is ≤ 2.

Now write h as a sum of monomials, and split every of these monomials in two ascending sequences as was done before, in Proposition 8. If there were two equal D_i's in one of the sequences in a monomial, then this monomial would vanish due to $z_1 z_2 z_1 = 0$. Therefore, if D_i appears twice in the monomial, then it must participate once in both sequences. But this means that we can write, up to sign, the monomial as $\cdots D_i^2 \cdots$. Then we use the fact that $D_i^2 D_j = -D_j D_i^2$ in order to take D_i^2 to the beginning of the monomial. Finally, observe that $D_i^2 D_j^2 = D_j^2 D_i^2$ for every i and j.

4. The graded identities of $G \otimes G$

We consider the tensor square of the Grassmann algebra G together with its natural 2-grading defined as $G \otimes G = (G_0 \otimes G_0 \oplus G_1 \otimes G_1) \oplus (G_0 \otimes G_1 \oplus G_1 \otimes G_0)$. Denote by I the ideal of 2-graded identities satisfied by $G \otimes G$. First we construct a model for the relatively free 2-graded algebra in the variety of 2-graded algebras defined by $G \otimes G$.

Let $a_i^{(0)}$, $b_i^{(0)}$, $c_i^{(0)}$, $d_i^{(0)}$ be commuting variables and $a_i^{(1)}$, $b_i^{(1)}$, $c_i^{(1)}$, $d_i^{(1)}$ be anticommuting ones, $i = 1, 2, \ldots$. We consider the free supercommutative algebra $K(Y; Z)$ freely generated by the sets $Y = \{a_i^{(0)}, b_i^{(0)}, c_i^{(0)}, d_i^{(0)}\}$ and $Z = \{a_i^{(1)}, b_i^{(1)}, c_i^{(1)}, d_i^{(1)}\}$ of even, respectively odd, variables. Set F the subalgebra of $K(Y; Z) \otimes K(Y; Z)$ generated by all elements of the form

$$Y_i = a_i^{(0)} \otimes b_i^{(0)} + a_i^{(1)} \otimes b_i^{(1)}, \quad Z_i = c_i^{(0)} \otimes d_i^{(1)} + c_i^{(1)} \otimes d_i^{(0)}.$$

Then $F = F_0 \oplus F_1$ is 2-graded algebra and the grading on it is the natural one, i.e., we consider Y_i as even and Z_i as odd variables.

LEMMA 13: *The 2-graded algebra F is isomorphic (as 2-graded algebra) to the relatively free algebra of countable rank $K(X)/I$ in the variety of 2-graded algebras defined by $G \otimes G$.*

Proof: The proof is rather straightforward. The homomorphism $K(X) \to F$ defined by $y_i \mapsto Y_i$ and $z_i \mapsto Z_i$ is surjective, and its kernel is equal to I, hence it is an isomorphism.

Now we need some elementary properties of the algebra $G \otimes G$. It is easy to see that its centre equals $G_0 \otimes G_0$.

LEMMA 14: *The polynomials $y_1y_2 - y_2y_1$, $z_1z_2z_3 + z_3z_2z_1$ are graded identities for $G \otimes G$. If $\operatorname{char} K = p > 2$ then the polynomial $y_1^p z_1 - z_1 y_1^p$ is also a graded identity for $G \otimes G$.*

Proof: The first polynomial is a 2-graded identity of $G\otimes G$ since $G_0\otimes G_0 \oplus G_1\otimes G_1$ is commutative algebra. The second polynomial is also a graded identity for $G\otimes G$ as direct calculation shows.

For the third, let $a \in G_0 \otimes G_0 \oplus G_1 \otimes G_1$; then $a = \sum(e_i \otimes f_i + g_i \otimes h_i)$ where e_i, $f_i \in G_0$, g_i, $h_i \in G_1$. Therefore

$$a^p = \sum(e_i^p \otimes f_i^p + g_i^p \otimes h_i^p) = \sum e_i^p \otimes f_i^p,$$

since $g_i^p = h_i^p = 0$ and the mixed terms vanish due to the binomial coefficients that are divisible by p. The element $\sum e_i^p \otimes f_i^p$ is central in $G \otimes G$, and this completes the proof.

Remark: Observe that according to the previous section, the last graded identity is not satisfied by the algebra $M_{1,1}(G)$. Hence it is not a consequence of the first two identities of the lemma.

We shall need the following relations satisfied by $G \otimes G$. Their deduction is straightforward; one can find it in [6], Lemma 2.2.

LEMMA 15: *The following equalities hold in the algebra $G \otimes G$:*

1. $z_1zz_1 = 0$, $z_1uz_1vz_1 = 0$, $z_1^2z_2^2 = z_2^2z_1^2$, $z_1^2z_2 = -z_2z_1^2$,
2. $t_1ut_2 = t_2ut_1 = 0$, $zt = -tz$,

for every t, t_1, $t_2 \in G_1 \otimes G_1$, z, z_1, $z_2 \in G_0 \otimes G_1 \oplus G_1 \otimes G_0$, u, $v \in G \otimes G$.

Observe that the graded identities of item (1) of the above lemma are consequences of the identities $y_1y_2 - y_2y_1$ and $z_1z_2z_3 + z_3z_2z_1$. This actually was proved in the previous section, since we established that they hold in the algebra $M_{1,1}$.

We already deduced that $T_2(M_{1,1}) \subseteq T_2(G\otimes G)$. Therefore the relatively free 2-graded algebra F is a homomorphic image of $\mathrm{Gen}(M_{1,1})$ and of L. Hence we have the following lemma.

LEMMA 16: *The algebra $B_2(F) = B_2(X)/(B_2(X)\cap I)$ is a homomorphic image of $B_2(L)$ (and of $B_2(\mathrm{Gen}(M_{1,1}))$ as well).*

Proof: For the proof observe that $T_2(M_{1,1}) \subseteq T_2(G\otimes G) = I$, therefore $B_2(X)/(B_2(X)\cap I)$ is a homomorphic image of $B_2(X)/(B_2(X)\cap T_2(M_{1,1}))$.

LEMMA 17: *Let $g_i(z_1, z_2, \ldots, z_n)$ be multilinear polynomials that are linearly independent modulo the T_2-ideal $T_2(M_{1,1})$. Then the polynomials*

$$y_1^{i_1} y_2^{i_2}\cdots y_k^{i_k} z_{n+1}^2 z_{n+2}^2\cdots z_{n+r}^2 g_i(z_1, z_2, \ldots, z_n)$$

are linearly independent modulo the T_2-ideal $T_2(M_{1,1})$.

Proof: Set $y_1 = E_1 + E_2 + \cdots + E_{i_1}$, $\ldots$, $y_k = E_{t+1} + \cdots + E_{t+i_k}$ for $t = i_1 + \cdots + i_{k-1}$, $z_{n+1} = D_{n+1} + D_{n+2}$, $\ldots$, $z_{n+r} = D_{n+2r-1} + D_{n+2r}$. This gives a non-zero factor, and then one applies the independence of g_i.

In order to proceed we need some combinatorics. Let $(i_1, i_2, \ldots, i_n)$ be a permutation of the symbols $\{1, 2, \ldots, n\}$, and assume that

$$\{1, 2, \ldots, n\} = A\cup B, \quad A\cap B = \emptyset.$$

One can consider colouring of these symbols in the colours A and B, which is the motivation for the terminology used. Then a pair (i_α, i_β), $1\le \alpha, \beta\le n$, forms a coloured inversion (with respect to the partition A, B) if $1\le\alpha<\beta\le n$, $i_\alpha > i_\beta$ and either $\alpha,\beta\in A$, or $\alpha,\beta\in B$. If q is the number of all coloured inversions in $(i_1, i_2, \ldots, i_n)$, then $(-1)^q$ is the coloured sign of this permutation with respect to the partition A, B. We shall consider the partitions A, B of $\{1, 2, \ldots, n\}$ as unordered pairs, i.e., we shall not distinguish (A, B) from (B, A). Then there are exactly 2^{n-1} partitions of $\{1, 2, \ldots, n\}$ including the trivial one. Obviously the coloured sign of the main (or trivial) permutation $(1, 2, \ldots, n)$ equals 1 for all partitions, since it does not contain (ordinary) inversions at all.

PROPOSITION 18: *Let $i = (i_1, i_2, \ldots, i_n)$ be a fixed permutation of $(1, 2, \ldots, n)$. Then the coloured sign of i equals either 1 for all partitions, or -1 for all partitions, or else 1 for 2^{n-2} partitions and -1 for the remaining 2^{n-2} partitions.*

Proof: The proof is an elementary combinatorial reasoning. It is easy to show that the transpositions $(t, t+2)$ change the coloured sign of every permutation, for every partition (A, B). In order to prove this fact, one considers all four possibilities for three consecutive symbols in the permutation: all belonging to A, i.e., $a_1a_2a_3$; or two to A and one to B, i.e., a_1a_2b, a_1ba_2, ba_1a_2. Here a_1, $a_2 \in A$, $b \in B$.

Using the above observation it is sufficient to prove the proposition only for permutations i such that $i_1 < i_3 < \cdots$ and $i_2 < i_4 < \cdots$.

We induct on n, $n = 1$ and 2 being obvious. For $n = 3$, we give a table with the coloured signs of all permutations of $\{1, 2, 3\}$ below. In particular, it shows that the statement is true for $n = 3$.

Permutations

Partitions		123	132	231	213	312	321
	123	+	−	+	−	+	−
	12; 3	+	+	−	−	+	−
	13; 2	+	+	−	+	−	−
	23; 1	+	−	+	+	−	−

Suppose $n > 3$ and that the assertion has been proved for all $m < n$. Let $i = (i_1, i_2, \ldots, i_n)$ and set $i' = (i_1, i_2, \ldots, i_{n-1})$ the permutation of $n-1$ symbols $\{1, 2, \ldots, n\} \setminus \{i_n\}$ obtained from i by deleting its last entry. Then the statement of the proposition holds for i' due to the induction. Note that either $i_n = n$ or $i_{n-1} = n$, since $i_1 < i_3 < \cdots$ and $i_2 < i_4 < \cdots$. Now we consider two cases for i_{n-1} and i_n.

Case 1: Assume $i_{n-1} < i_n$, hence $i_n = n$. If (A, B) is a partition of $\{1, 2, \ldots, n\} \setminus \{i_n\}$ we form two partitions of $\{1, 2, \ldots, n\}$. These are $(A \cup \{i_n\}, B)$ and $(A, B \cup \{i_n\})$. The coloured signs of i with respect to these two partitions will be the same as the coloured sign of i' with respect to (A, B).

Case 2: Let $i_{n-1} > i_n$, then $i_{n-1} = n$. Hence i_{n-1} forms inversion with i_n only. Set $i'' = (i_1, i_2, \ldots, i_{n-2}, i_n)$. Then our statement holds for i''. Let (C, D) be a partition of $\{1, 2, \ldots, n-1\}$ and let ϵ be the coloured sign of i'' with respect to (C, D). Then we form the partitions $(C \cup \{n\}, D)$ and $(C, D \cup \{n\})$ of $\{1, 2, \ldots, n\}$. In one of them $i_{n-1} = n$ and i_n belong to different sets of the partition, hence the coloured sign of i with respect to this partition will be ϵ. Analogously, in the other partition $i_{n-1} = n$ and i_n are in the same set, and since they do form an inversion, this yields coloured sign $-\epsilon$.

Now both cases are dealt with, and the proof of the proposition is complete.

Corollary 19: *The coloured sign of $\sigma = (n, n-1, \ldots, 2, 1)$ equals 1 for all partitions when $n \equiv 1 \pmod 4$ and -1 when $n \equiv 3 \pmod 4$. If n is even, then the coloured sign of σ equals 1 for 2^{n-2} partitions and -1 for the rest.*

Proof: Suppose $\{1, 2, \ldots, n\} = A \cup B$, $|A| = a$, $|B| = b$, $a + b = n$. Then there will be $q = a(a-1)/2 + b(b-1)/2$ coloured inversions in σ. One has that

$$q = (a^2 + b^2 - a - b)/2 = (n^2 - n)/2 - ab = n(n-1)/2 - ab.$$

If n is odd then one of a and b is even, and this yields the statement for $n \equiv 1$ and $3 \pmod 4$.

If $n = 2m$ is even, using that $\sum_{j=1}^{n}(-1)^j\binom{n}{j} = 0$ we obtain the equality $\sum_{j=0}^{m-1}(-1)^j\binom{n}{j} + (1/2)(-1)^m\binom{n}{m} = 0$. (Note that in the last case $\binom{n}{m} = \binom{2m}{m}$ is always even.)

PROPOSITION 20: *Let $i = (i_1, i_2, \ldots, i_n)$ be a permutation of the symbols $(1, 2, \ldots, n)$ and suppose $i_1 < i_3 < \cdots$ and $i_2 < i_4 < \cdots$. If $i \neq (1, 2, \ldots, n)$, then the coloured sign of i equals 1 for 2^{n-2} partitions and -1 for the remaining 2^{n-2} partitions of $\{1, 2, \ldots, n\}$.*

Proof: It follows from the proof of Proposition 18. We induct on n. In the first case considered there, due to the inductive assumption and to $i \neq (1, 2, \ldots, n)$ we obtain 2^{n-2} times coloured sign 1, and 2^{n-2} times coloured sign -1. The same holds for the second case.

Remark: Note that, in fact, in the previous statements we gave a combinatorial description of the Meson algebras; see, for example, [9], pp. 115 and 264–272. Of course, our goal was not the description of these algebras but it came "for free."

COROLLARY 21: *The multilinear monomials*

$$m_{ij} = z_{i_1} z_{j_1} z_{i_2} z_{j_2} \cdots z_{i_m} \widehat{z_{j_m}}$$

where $i_1 < i_2 < \cdots < i_m$, $j_1 < j_2 < \cdots < j_{m-1} < j_m$, are linearly independent modulo the graded identities of the algebra $G \otimes G$. Here, if the degree of the monomial is odd, z_{j_m} is missing.

Proof: Suppose on the contrary that they are linearly dependent, and that $\sum \alpha_{ij} m_{ij} = 0$. Then the latter will be a graded identity for $G \otimes G$. Due to the homogeneity we can suppose that all m_{ij} are monomials in z_1, z_2, $\ldots$, z_k. Suppose that $z_1 z_2 \cdots z_{k-1} z_k$ participates in such a linear combination with nonzero coefficient α. We choose a partition $A \cup B$ of the set $\{1, 2, \ldots, k-1, k\}$ and let $z_i \mapsto e_i \otimes 1$ whenever $i \in A$, $z_j \mapsto 1 \otimes e_j$, $j \in B$. The evaluation of the combination will be 0. Sum up the evaluations for all partitions A and B. The elements of $G_1 \otimes G_0$ anticommute, and the same holds for $G_0 \otimes G_1$. The elements of $G_0 \otimes G_1$

commute with those of $G_1 \otimes G_0$. Thus one applies the proposition above and obtains that $2^h\alpha = 0$ for some positive integer h. This is a contradiction since $\operatorname{char} K \neq 2$.

COROLLARY 22: *Let $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in B_2(M_{1,1}) \equiv B_2(L)$ be a graded polynomial. (Here we use the letters y_i for E_i and z_i for D_i; see the definition of L in the previous section.) Then modulo the ideal I of the graded identities of $G \otimes G$ it equals a polynomial of the form*

$$y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_m^{\alpha_m} z_{i_1}^2 z_{i_2}^2 \cdots z_{i_k}^2 g_j(z_{j_1}, z_{j_2}, \ldots, z_{j_l})$$

where $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_l\} = \emptyset$, $i_1 < i_2 < \cdots < i_k$, and the polynomial g_j is multilinear. If $\operatorname{char} K = p > 0$, we impose further $\alpha_i < p$, $i = 1, \ldots, m$.

Furthermore, if the multilinear polynomials g_j are linearly independent modulo I, then the above polynomials are linearly independent as well.

Proof: The proof is the same as that for $B_2(M_{1,1})$. Note that if $a \in G_1 \otimes G_1$, then $a^p = 0$.

THEOREM 23: *The ideal of the 2-graded identities of the algebra $G \otimes G$ is generated by the polynomials $y_1y_2 - y_2y_1$, $z_1z_2z_3 + z_3z_2z_1$, and if $\operatorname{char} K = p > 2$, by $y_1^p z_1 - z_1 y_1^p$.*

Proof: We already established that these polynomials are indeed graded identities for $G \otimes G$. Since the first two of them form a basis for the graded identities of $M_{1,1}$, we can work in the relatively free algebra determined by them, i.e., in $\operatorname{Gen}(M_{1,1})$. As we showed, it is sufficient to consider only the polynomials such that the even variables appear in them in commutators only. But in this case the last corollary yields the statement of the theorem.

As a corollary of the last theorem we obtain another proof of the coincidence of the T-ideals of the algebras M_{11} and $G \otimes G$ over a field of characteristic 0. We note that this is a "standard" fact. Its known proofs use either the structure theory of T-ideals (see [10], p. 24) or the description of the basis of $T(G \otimes G)$ given by Popov, [12] (see [5], Theorem 2), or other deep methods and results (see [15], Theorem 4.7). The proof we give is elementary.

COROLLARY 24: *Let $\operatorname{char} K = 0$. Then the algebras M_{11} and $G \otimes G$ are PI equivalent. In other words, $T(M_{11}) = T(G \otimes G)$.*

Proof: Let A and B be two 2-graded algebras over a field K, $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ being the decompositions of A and B into even and odd parts.

If A and B satisfy the same 2-graded polynomial identities, then obviously they satisfy the same ordinary identities. As we showed in the case of characteristic 0, the 2-graded identities of M_{11} and of $G \otimes G$ follow from the identities

$$y_1y_2 - y_2y_1 = 0, \qquad z_1z_2z_3 + z_3z_2z_1.$$

Therefore the algebras M_{11} and $G \otimes G$ satisfy the same ordinary polynomial identities when $\operatorname{char} K = 0$.

OPEN PROBLEMS.

1. Determine the ordinary identities satisfied by the algebras $M_{1,1}$ and $G \otimes G$ over an infinite field of characteristic $p > 2$. Or (weaker): determine the difference between the T-ideals of these two algebras.
2. Probably the determination of the weak identities for the algebra $M_{1,1}$ would help in finding a basis of the identities for this algebra. (Recall that $f \in K\langle X\rangle$ is weak identity for $M_{1,1}$ if it vanishes under substitutions of matrices $f_i(e_{11}+e_{22})+g_ie_{12}+h_ie_{21}$ where $f_i \in G_0$ and g_i, $h_i \in G_1$. Sometimes these are called matrices with supertrace zero.) It seems plausible that the weak identities of $M_{1,1}$ follow from the weak identities $[x_1, x_2, x_3] = 0$ and $[x_1, x_2][x_1, x_3][x_1, x_4] = 0$, at least if $\operatorname{char} K \neq 2$ or 3. O. M. Di Vincenzo and R. La Scala [19] communicated to us that this is true if $\operatorname{char} K = 0$.
3. Another interesting problem related to the algebras considered in this paper seems to be the following. Describe the possible 2-gradings of the algebras M_2, M_{11}, $G \otimes G$ in terms of the polynomial identities they satisfy. Note that this could help in resolving the problem of coincidence of $T(M_{11})$ and $T(G \otimes G)$ in positive characteristic.
4. What further information about the identities (ordinary and 2-graded) of M_{11} and $G \otimes G$ can be deduced? In this direction, what information can the codimension sequences and, more important, the Hilbert series of the corresponding relatively free algebras, yield? Note that computing the graded codimensions and Hilbert series of these relatively free algebras is a simple technical question, since we provided linear bases of these algebras.

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